COMMENT ON "SIMPLEST POSSIBLE SELF-ORGANIZED CRITICAL SYSTEM"

Flyvbjerg [1] discusses a simplification of the well-known sandpile automaton models [2] that contains only two degrees of freedom, namely the number of particles in the pile and the number of particles in the avalanche. This system is characterized by two parameters, the number of dynamical sites N and the number of absorbing sites M. The author finds a mean field distribution of avalanche sizes $P(\tau) \sim \tau^{-3/2}$ known from a number of related systems (see refs. [10-13] in [1]). Life, however, can be more dangerous than in mean field theory! This follows from the solution of the full dynamics we report here. In the scaling limit $N \gg 1$ and for small effective absorption $\mu \equiv M/\sqrt{N}$, we obtain the asymptotic avalanche size distribution

$$P(\tau) \sim \begin{cases} \tau^{-\frac{3}{2}} & (\tau \lesssim N) \\ \left(1+\frac{2}{\sqrt{-k}}\frac{\mu\tau}{N}\right) \times & \\ \times \exp\left[-\frac{2k+1}{k}\left(\frac{\mu\tau}{N}\right)^2 - 2\sqrt{-k}\frac{\mu\tau}{N}\right] & (\tau \gtrsim N) \end{cases} \quad (1)$$

with $k = \ln(8\pi\mu^2)$, in excellent agreement with the numerical result shown in Fig. 1(a). While the power law still holds for avalanches of size $\tau \lesssim N$, surprising effects arise for large avalanches: the number of avalanches of size $N \lesssim \tau \lesssim N/\mu$ is approximately constant and therefore drastically *increased* over the power law behavior. Only very large avalanches of size $\tau \gtrsim N/\mu$ are exponentially suppressed. Hence, the avalanche distribution is dominated by large events, and only a small fraction of the avalanches falls into the power law regime $P(\tau) \sim \tau^{-3/2}$.

For large systems, the time evolution of avalanches is governed by coupled Langevin equations in the variables x and $y > 0$ (describing the rescaled number of particles in the pile and in the avalanche, respectively), or by the equivalent Fokker-Planck equation (7) of [1].

For small values of μ, a *typical* avalanche starting from a pile of size $x = x_0$ either terminates after a time $\tau \lesssim N$ (while it is still dominated by the diffusion term) or grows up to a maximum size $y_{\max} \approx x_0$. The subsequent decay of these large avalanches evolves around the trajectory $\bar{x}(\tau) = 0$, $\bar{y}(\tau) = x_0 - \mu\tau/N$ with fluctuations δx, δy of order 1. Hence, a large avalanche is typically trapped in the system for a time $\tau = N x_0/\mu$, only rare fluctuations may lead to an earlier end. To make this picture more quantitative, we use the representation of the avalanche dynamics as a perturbed harmonic oscillator with slowly moving center; see Eq. (11) of [1]. At times $\tau \lesssim N$, the harmonic potential can be neglected over the diffusion term, and the avalanche time distribution $P(\tau) \sim \tau^{-3/2}$ is simply the probability of first return of a diffusion path to the boundary $y = 0$ [3]. At later times, this return probability is dominated by the ground state of the perturbed harmonic oscillator. For fixed center, this problem is well known in the context of two-dimensional electron gases in a magnetic field [4] and the quantum Hall effect. In the adiabatic limit $\mu \ll 1$, one obtains the approximate time distribution

$$P^{\rm as}_{x_0}(\tau) = \frac{4\bar{y}(\tau)e^{-2\bar{y}^2(\tau)}}{\sqrt{2\pi}} \exp\left[-\frac{e^{-2\bar{y}^2(\tau)} - e^{-2x_0^2}}{\sqrt{2\pi}\mu}\right]$$

of large avalanches starting from a pile of size $x_0 \gtrsim 1$.

The next step is to calculate the steady-state distribution $p(x)$ of pile sizes *between* avalanches, which is dynamically coupled to the distribution $\tilde{p}(\bar{y})$ *during* large avalanches. Solving the resulting system of two first-order differential equations, one has for small μ (up to a normalization constant)

$$p(x) \sim \exp\left(-\frac{1}{\sqrt{2\pi}\mu}e^{-2x^2} - 2x^2\right), \quad (2)$$

which fits the numerical data of Fig. 1(b) perfectly. In contrast to the conjecture of [1], $p(x)$ is not purely Gaussian. Finally, the time distribution of large avalanches in Eq. (1) is given by $\int s(x_0)p(x_0)P^{\rm as}_{x_0}(\tau)\mathrm{d}x_0$, where $s(x_0) \sim x_0$ is the probability that a large avalanche starts from a pile of size x_0. This integral can be evaluated in a saddle-point approximation.

To summarize, system-wide fluctuations beyond the reach of mean field theory can dominate even the simplest self-organized critical dynamics. It remains to be seen whether similar effects exist in more realistic models.

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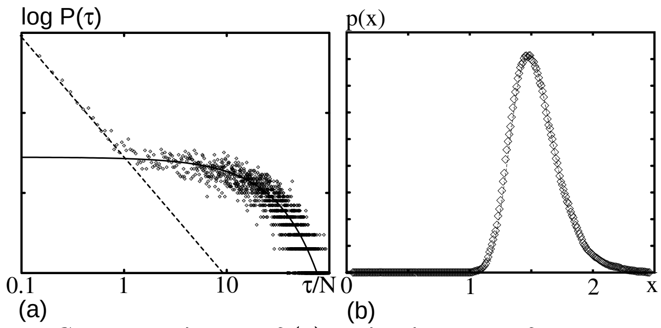

FIG. 1. Distribution of (a) avalanche sizes τ for $\mu = 0.01$ and (b) pile sizes x between avalanches for $\mu = 0.005$, compared to the predictions of Eqns. (1) and (2), respectively.